Name: ______________________________

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**Math 10550, EXAM II**
**October 13, 3016**

- The Honor Code is in effect for this examination. All work is to be your own.
- No calculators.
- The exam lasts for 1 hr. and 15 min.
- Be sure that your name is on every page in case pages become detached.
- Be sure that you have all 15 pages of the test.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

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| 1. | (a) | (b) | (c) | (d) | (e) |
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| **Multiple Choice** | ______________ |
| 11. | ______________ |
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## Multiple Choice

**1.**(6 pts.) Find $y'$, if

$$x^4 + x^3y + 5xy^2 = 8.$$

(a) $\dfrac{-(4x^3+3x^2y+5y^2)}{10xy}$

(b) $\dfrac{-(4x^3+3x^2y+5y^2)}{x^3}$

(c) $\dfrac{-(4x^3+3x^2y)}{x^3+10xy}$

(d) The derivative does not exist.

(e) $\dfrac{-(4x^3+3x^2y+5y^2)}{x^3+10xy}$

**Solution:** We differentiate both sides with respect to $x$

$$\frac{d(x^4+x^3y+5xy^2)}{dx} = \frac{d(8)}{dx}$$
$$4x^3+3x^2y+x^3\frac{dy}{dx}+5y^2+10xy\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{-4x^3-3x^2y-5y^2}{x^3+10xy}$$

**2.**(6 pts.) A particle is moving in a straight line along a horizontal axis with a position function given by

$$s(t) = t^2 - 4t + 4,$$

where distance is measured in feet and time is measured in seconds. What is the distance travelled by the particle in the time period $1 \le t \le 4$ seconds?

(a) 0 feet

(b) 5 feet

(c) 8 feet

(d) 2 feet

(e) 3 feet

**SOLUTION:**

Recall that the distance is given by how much has the particle travelled, thus we need to look for any point where our particle changed direction. To do so we consider where (if any) does the derivative of $s$ change signs. We compute

$$s'(t) = 2t - 4$$

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Which is zero at $t = 2$ and since it is an increasing line we know $s'(t) < 0$ for $t < 2$ and $s'(t) > 0$ for $t > 2$. Thus it changes direction on $[1, 4]$ only at $t = 2$. Hence the total travelled distance (since the particle is moving in a straight line along a horizontal axis), in feet, is given by

$$D = |s(4) - s(2)| + |s(2) - s(1)| = |4 - 0| + |0 - 1| = 5.$$

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**3.**(6 pts.) A right triangle has base $x$ feet and height $y$ feet. If the base increases at 2 ft/second, and the height increases at 1 ft/second, find the rate of change in the area of the right triangle when $x = 8$ and $y = 5$.

| | | | | | |
|---|---|---|---|---|---|
| (a) | 9 ft$^2$/second | (b) | 18 ft$^2$/second | (c) | 10.5 ft$^2$/second |
| (d) | 2 ft$^2$/second | (e) | $-1$ ft$^2$/second | | |

**SOLUTION:**

The formula for the area A of a triangle with base $x$ in feet and height $y$ in feet is given by $A = \frac{1}{2}xy$. Differentiating with respect to $t$, we get that

$$\frac{dA}{dt} = \frac{1}{2}\frac{dx}{dt}y + \frac{1}{2}x\frac{dy}{dt}$$

Plugging in the values given in the problem, we see that

$$\begin{aligned}\frac{dA}{dt} &= \frac{1}{2}(2)(5) + \frac{1}{2}(8)(1) \\ &= 9\end{aligned}$$

**4.**(6 pts.) Suppose $f$ is differentiable and $-2 \leq f'(x) \leq 1$ for all $x$ and $f(2) = 3$. What are the minimum and maximum possible values for $f(5)$?

| | | | | | |
|---|---|---|---|---|---|
| (a) | $-3 \leq f(5) \leq 0$ | (b) | $3 \leq f(5) \leq 6$ | (c) | $-3 \leq f(5) \leq 6$ |
| (d) | $-10 \leq f(5) \leq 10$ | (e) | $-5 \leq f(5) \leq 4$ | | |

**SOLUTION:**

Since $f$ is differentiable we can apply the Mean Value Theorem on $[2, 5]$ to get the existence of an $x \in (2, 5)$ such that

$$f'(x) = \frac{f(5) - f(2)}{5 - 2}$$

And by assumption (on $f'$) we have,

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$$-2 \leq \frac{f(5)-3}{3} \leq 1$$

$$\implies -3 \leq f(5) \leq 6$$

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**5.**(6 pts.) Use linear approximation of $f(x) = \sqrt[3]{x}$ at $a = -8$ to estimate $\sqrt[3]{-8.12}$.

(a) $-2.04$ (b) $-1.99$ (c) $-1.8$ (d) $-2.2$ (e) $-2.01$

**SOLUTION:**

Using the linear approximation formula, we see that

$$L(x) = f(a) + f'(a)(x - a)$$

The derivative of $f$ is $f'(x) = \dfrac{1}{3x^{2/3}}$. We see that $f(-8) = -2$ and

$$\begin{aligned} f'(-8) &= \frac{1}{3}\frac{1}{(-2)^2} \\ &= \frac{1}{12} \end{aligned}$$

So

$$L(x) = -2 + \frac{1}{12}(x + 8)$$

Thus, the approximation is

$$\begin{aligned} L(-8.12) &= -2 + \frac{1}{12}(-8.12 + 8.12) \\ &= -2 + -.01 \\ &= -2.01 \end{aligned}$$

**6.**(6 pts.) Find the linearization of the function $f(x) = \sin^2(x)$ at $a = \dfrac{\pi}{4}$.

(a) $\dfrac{x}{2} + \dfrac{1}{2} - \dfrac{\pi}{4}$ (b) $x + \dfrac{1}{2} - \dfrac{\pi}{4}$ (c) $x + \dfrac{1}{\sqrt{2}} - \dfrac{\pi}{4}$

(d) $-\dfrac{10}{4}x - \dfrac{1}{4}$ (e) $\dfrac{1}{2}x + \dfrac{3}{2}$

**SOLUTION:**

Recall that the "Linearization" of a function at a certain point $a$ is just its tangent line at the given point; and it is given by

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$$L(x) = f'(a)(x-a) + f(a).$$

Since $f'(x) = 2(\sin(x))\cos(x)$ we get $f'(a) = f'(\frac{\pi}{4}) = 2(\sin(\frac{\pi}{4}))\cos(\frac{\pi}{4}) = 2\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} = 1$. Also $f(a) = f(\frac{\pi}{4}) = \left(\sin(\frac{\pi}{4})\right)^2 = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$. Hence,

$$L(x) = x - \frac{\pi}{4} + \frac{1}{2}.$$

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**7.**(6 pts.) Which of the following gives a complete list of the critical numbers/points of the function

$$f(x) = 3x^{2/3} \cdot (x+1)^3$$

(a) $x = \dfrac{-2}{11},\ -1$ (b) $x = \dfrac{2}{11},\ -1$ (c) $x = 0,\ \dfrac{-2}{11},\ -1$

(d) $x = \dfrac{2}{11},\ 1$ (e) $x = -1, 0$

**SOLUTION:**

To be a critical number, you must either be a number $a$ in the domain of $f$ such that $f(a) = 0$ OR a number $a$ in the domain of $f$ such that $f'$ does not exist at $a$. We observe that $f$ is defined for every real number. Its derivative is given by

$$\begin{aligned} f'(x) &= \frac{2(x+1)^3}{x^{1/3}} + 9x^{2/3}(x+1)^2 \\ &= \frac{2(x+1)^3 + 9x(x+1)^2}{x^{1/3}} \end{aligned}$$

We see that $f'$ is not defined at $a = 0$ and 0 is in the domain of $f$. So 0 is a critical number. Now, to find the rest of the critical values, we set $f' = 0$. We see this reduces to the equation

$$2(x+1)^3 + 9x(x+1)^2 = 0$$

This gives

$$(x+1)^2(2(x+1) + 9x) = 0$$

So $x = -1$ is a solution and solving

$$2(x+1) + 9x = 11x + 2 = 0$$

tells us that $x = -2/11$ is another solution. Thus, the full set of critical numbers is $x = 0, -2/11, -1$.

**8.**(6 pts.) Let $f(x) = 4x^2 - 4x + 4$. Find the absolute maximum and absolute minimum of $f$ on the interval $[0, 2]$. (That is find the maximum and minimum value of $f(x)$ on the given interval).

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(a) Max value = 4, Min value = 3

(b) Max value = 12, Min value = 4

(c) Max value = 12, No Minimum value exists

(d) Max value = 6, Min value = 4

(e) Max value = 12, Min value = 3

**SOLUTION:**

First we compute the derivative, $f'(x) = 8x - 4$. Setting $f'(x) = 0 = 8x - 4$, we see that $x = 1/2$ is the only critical number (note that $f'(x)$ is defined for every real number). The absolute maximum and absolute minimum must occur at a critical number or at an endpoint of the interval. So we simply evaluate $f$ at each of these points. We see that

$$f(0) = 4$$

$$f(2) = 4 \cdot 4 - 4 \cdot 2 + 4 = 12$$

and

$$f(1/2) = 4(1/4) - 4(1/2) + 4 = 3$$

From our computations, we see that $f$ has max value 12 and min value 3.

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**9.**(6 pts.) Let

$$f(x) = \frac{x^3}{3} - 2x^2 - 12x + 17.$$

On which of the intervals given below is the graph of $f$ **both** decreasing and concave down (on the entire interval)?

(a) $(2, 6)$ (b) $(-2, 6)$ (c) $(-2, 2)$ (d) $(-\infty, 2)$ (e) $(6, \infty)$

**SOLUTION:**

We are looking for an interval on which both $f'$ and $f''$ are negative. We compute,

$$f'(x) = x^2 - 4x - 12 = (x - 6)(x + 2)$$

Notice that for any $x \in (-\infty, -2) \cup (6, \infty)$ we have $f'(x) > 0$. Also for $x \in (-2, 6)$ we have $f'(x) < 0$. We also compute,

$$f''(x) = 2x - 4.$$

Which is an increasing line with $x$-intercept at $x = 2$ and hence $f''(x) < 0$ for $x \in (-\infty, 2)$ and $f''(x) > 0$ for $x \in (2, \infty)$. By intersecting the two intervals of interest, i.e. those intervals for which both are negative, we get the interval $(-2, 2)$.

**10.**(6 pts.) Consider the function $f(x) = x^4 - 8x^3 + 5$. Which of the following statements is true?

(a) $f$ has a local minimum at $x = 6$, no local maximum, and points of inflection at $x = 0$ and 4.

(b) $f$ has a local minimum at $x = 6$, a local maximum at $x = 0$, and points of inflection at $x = 0$ and 4.

(c) $f$ has a local maximum at $x = 0$, no local minimum, and a point of inflection at $x = 4$.

(d) $f$ has local minima at $x = 0$ and 6, no local maximum, and a point of inflection at $x = 4$.

(e) $f$ has a local minimum at $x = 6$, no local maximum, and points of inflection at $x = 0, 4$ and $-4$.

**SOLUTION:**

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We begin by first computing the derivative and second derivative. We compute

$$\begin{aligned} f'(x) &= 4x^3 - 24x^2 \\ &= 4x^2(x-6) \end{aligned}$$

and

$$\begin{aligned} f''(x) &= 12x^2 - 48x \\ &= 12x(x-4) \end{aligned}$$

Now, we see that the zeros of $f''(x)$ are $0, 4$. When $x < 0$, we see that $f''(x)$ is positive by choosing any test point such as $-1$. We see that

$$\begin{aligned} f''(-1) &= 12(-1)(-1-4) \\ &= 60 \\ &> 0 \end{aligned}$$

When $0 < x < 4$, we see that $f''(x)$ is negative by choosing a test point such as 1. We see that

$$\begin{aligned} f''(1) &= 12(-3) \\ &= -36 \\ &< 0 \end{aligned}$$

When $x > 4$, we see that $f''(x)$ is positive by choosing a test point such as 5. We see that

$$\begin{aligned} f''(5) &= 5(12 \cdot 5 - 48) \\ &= 5 \cdot 12 \\ &= 60 \\ &> 0 \end{aligned}$$

Our computations show that $f''$ changes signs at $x = 0$ and $x = 4$. This tells us that 0 and 4 are both points of inflection. Now, the zeros of the derivative (i.e, the critical numbers of $f$) are 0 and 6. We see that $f''(6) > 0$ and so the second derivative test tells us that at $x = 6$, there is a local minimum. By choosing test points such as $-1$ and 1, we see that $f'(1) = -20 < 0$ and $f'(-1) = -28 < 0$. Thus, at the point $x = 0$, there is neither a local minimum nor a local maximum. From all that we have shown, we see that $a$ must be the correct solution. Note that this solution is given for completeness but we did not have to do all this work to arrive at the correct answer. For example, once we saw that the only points of inflection occurred at $x = 0$ and $x = 4$, we could eliminate three of the answer choices. Thus, we were left to only decide what happened at $x = 0$.

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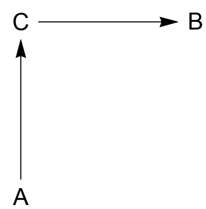
**Partial Credit**

You must show your work on the partial credit problems to receive credit!

**11.**(11 pts.) Pedestrian $A$ is walking towards the intersection $C$ of two streets intersecting at a right angle. Pedestrian $B$ is walking away from intersection $C$. Pedestrian $A$ is going North at 2 mph, and Pedestrian B is going East at 3 mph. How fast is the distance from Pedestrian A to Pedestrian B changing when Pedestrian A is 4 miles South of intersection $C$, and Pedestrian B is 3 miles East of intersection C.




**SOLUTION:**

Let us denote by $z(t)$ the distance at any time between the pedestrians, by $y(t)$ the distance between $A$ and the intersection and by $x(t)$ the distance between $B$ and the intersection. We are given,

$$\frac{dy}{dt} = -2 \quad ; \quad \frac{dx}{dt} = 3.$$

And we want to know

$$\left.\frac{dz}{dt}\right|_{y=4,x=3}.$$

We know by the Pythagorean Theorem that $z^2(t) = y^2(t) + x^2(t)$ and differentiating with respect to time in both sides gives

$$2z(t)\frac{dz}{dt} = 2y(t)\frac{dy}{dt} + 2x(t)\frac{dx}{dt}$$

$$\Longrightarrow \frac{dz}{dt} = \frac{1}{z(t)}\left[y(t)\frac{dy}{dt} + x(t)\frac{dx}{dt}\right].$$

We also know (again by the Pythagorean theorem) that if $y = 4$ and $x = 3$, then $z = 5$, so that in mph

$$\left.\frac{dz}{dt}\right|_{y=4,x=3} = \frac{1}{5}\left[4(-2) + 3(3)\right] = \frac{1}{5}$$

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**12.**(13 pts.) Consider the function

$$f(x) = x^3 + x - \frac{1}{x},$$

with domain $(0, \infty)$. With this restriction on the domain, show that the equation $f(x) = 0$ has one and exactly one real solution for $x \in (0, \infty)$. Identify the theorem(s) you are using and show the validity of the required conditions to apply the theorems are true to gain full credit.

**SOLUTION:**

We first note that $f$ is continuous on $(0, \infty)$. Thus, we may attempt to use the Intermediate Value Theorem. Now, we see that $f(1/2) = 1/8 + 1/2 - 2 = 5/8 - 2 < 0$. Also, we see that $f(2) = 8 + 2 - 1/2 = 10 - 1/2 > 0$. Thus, $f$ changes signs between $1/2$ and $2$. Since $f$ is also continuous on $(1/2, 2)$, the Intermediate Value Theorem tells us that there is a root of $f$ somewhere in $(1/2, 2)$.

To justify that there is at most one solution, we invoke the Mean Value Theorem or Rolle's Theorem. If there were more than one root of $f$, then there must be at least two.Call them $a$ and $b$ and say $b < a$. As $f$ is continuous and differentiable on all of $(0, \infty)$, we have that $f$ must be continuous on $[b, a]$ and differentiable on $(b, a)$. The Mean Value Theorem or Rolle's Theorem both imply that there is some $c$ in $(b, a)$ such that

$$\begin{aligned} f'(c) &= \frac{f(b) - f(a)}{b - a} \\ &= 0, \end{aligned}$$

Now $f'(x) = 3x^2 + 1 + \frac{1}{x^2}$. We see that $f'(x) > 0$ for all values of $x$ in $(0, \infty)$, since $f'(x) = 1 + (sum of squares) \geq 1$, since a sum of squares is always non-negative. Since the derivative is always strictly greater than 0 on $(0, \infty)$ and so in particular on $(b, a)$. So we have a contradiction.

Thus the assumption that there is more than one solution must be false.

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**13.**(16 pts.) Let $f(x) = x - \sin(2x)$, with domain $(-\frac{\pi}{2}, \frac{\pi}{2})$.

(a) Find the critical numbers/points for $f$.

**SOLUTION:**

Since $f'$ is continuous we only look for when does $f'(x) = 1 - 2\cos(2x) = 0$. This happens when $\cos(2x) = \frac{1}{2}$, which implies that $2x = \pm\frac{\pi}{3} \implies x = \pm\frac{\pi}{6}$.

(b) Find the intervals where $f$ is increasing and decreasing. (justify your answer)

**SOLUTION:**

Knowing that $f$ has critical points at $\pm\frac{\pi}{6}$ we select points from the three new formed intervals $(-\frac{\pi}{2}, -\frac{\pi}{6})$, $(-\frac{\pi}{6}, \frac{\pi}{6})$ and $(\frac{\pi}{6}, \frac{\pi}{2})$, say $-\frac{\pi}{4}$, 0 and $\frac{\pi}{4}$ respectively. We see that $f'(\pm\frac{\pi}{4}) = 1 > 0$ and that $f'(0) = 1 - 2 = -1 < 0$. So that by the first derivative test $f$ is increasing on $(-\frac{\pi}{2}, -\frac{\pi}{6}) \cup (\frac{\pi}{6}, \frac{\pi}{2})$ and decreasing on $(-\frac{\pi}{6}, \frac{\pi}{6})$.

(c) Classify the critical points as local maxima or local minima or neither and justify your conclusions in each case.

**SOLUTION:**

By what we just found in part (b) and the first derivative test, we know that $f$ has a local maximum at $-\frac{\pi}{6}$ and a local minimum at $\frac{\pi}{6}$. Moreover the second derivative test gives $f"(-\frac{\pi}{6}) = 4\sin(-\frac{\pi}{3}) < 0$ implies it is a local maximum while $f"(\frac{\pi}{6}) = 4\sin(\frac{\pi}{3}) > 0$ implies it is a local minimum.

(d) Find the intervals where $f$ is concave up and concave down, and find all inflection points.

Note that $f''(x) = 4\sin(2x)$ is zero on $(-\frac{\pi}{2}, \frac{\pi}{2})$ only when $x = 0$. And by parity we know $f''$ is negative on $(-\frac{\pi}{2}, 0)$ and positive on $(0, \frac{\pi}{2})$. This ensures that 0 is in fact an inflection point and that $f$ is concave down on the former interval and concave up on the latter.

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ROUGH WORK

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**Math 10550, EXAM II**
**October 13, 3016**

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**Please do NOT write in this box.**

**Multiple Choice** ______________

11. ______________

12. ______________

13. ______________

Total ______________